

On sums of tensor and fusion multiplicities

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Abstract

The total multiplicity in the decomposition into irreducibles of the tensor product $\lambda \otimes \mu$ of two irreducible representations of a simple Lie algebra is invariant under conjugation of one of them $\sum_{\nu} N_{\lambda\mu}^{\nu} = \sum_{\nu} N_{\bar{\lambda}\mu}^{\nu}$. This also applies to the fusion multiplicities of affine algebras in conformal WZW theories. In that context, the statement is equivalent to a property of the modular S matrix, viz $\Sigma(\kappa) := \sum_{\lambda} S_{\lambda\kappa} = 0$ if κ is a complex representation. Curiously, this vanishing of $\Sigma(\kappa)$ also holds when κ is a quaternionic representation. We provide proofs of all these statements. These proofs rely on a case-by-case analysis, maybe overlooking some hidden symmetry principle. We also give various illustrations of these properties in the contexts of boundary conformal field theories, integrable quantum field theories and topological field theories.

Introduction

In the course of investigations of algebraic features of conformal theories, we have encountered a seemingly unfamiliar property of sums of tensor product or fusion multiplicities of irreducible representations of simple or affine Lie algebras, and an associated property of the modular S -matrix in the affine algebra case. Let $N_{\lambda\mu}^\nu$ the multiplicity of the irrep of weight ν in the tensor product of those of weights λ and μ . (Notations will be presented with more care in the following section.) It is a commonplace to say that $N_{\lambda\mu}^\nu = N_{\bar{\lambda}\bar{\mu}}^{\bar{\nu}}$ and that $N_{\lambda\mu}^\nu = N_{\bar{\nu}\bar{\mu}}^{\bar{\lambda}}$, where $\bar{\lambda}$ is the complex conjugate weight of λ , and hence that $\sum_\nu N_{\lambda\mu}^\nu$ is invariant under the simultaneous conjugation of λ and μ . We claim that the latter sum is also invariant under a single conjugation $\lambda \rightarrow \bar{\lambda}$: $\sum_\nu N_{\lambda\mu}^\nu = \sum_\nu N_{\bar{\lambda}\mu}^\nu$ (Theorem 1). The paper consists of variations on that theme.

Here is the layout of the paper. The main results are presented in Sect 1 as a sequence of four theorems. Theorem 1 is the above property for tensor product multiplicities and Theorem 2 deals with the same property for fusion coefficients within affine algebras. Theorem 3 and 4 assert that the sum $\Sigma(\kappa) := \sum_\lambda S_{\lambda\kappa}$ vanishes if κ is a complex (Th 3) or quaternionic (Th 4) representation. Proofs of Theorems 1, 2, 3 and 4 will be given in Sect 2, 3, 4 and 5, respectively. Sect 6 is a short discussion of cancellations of $\Sigma(\kappa)$ that may also occur when κ is real, Sect 7 shows what may happen in finite groups, and Sect 8 presents a few applications or illustrations of our properties in various contexts, together with some final comments. Appendices gather lengthy details of our proofs, and some useful tables.

1 The main results

Let \mathfrak{g} be a finite dimensional simple Lie algebra of rank n . Each of its finite dimensional irreducible representations (irreps) is labelled by a highest weight (h.w.) λ . By a small abuse of notation, we refer to that representation as representation λ . Throughout this paper, we shall denote $[\lambda]$ the weight system of irrep λ . Let $N_{\lambda\mu}^\nu$ denote the multiplicity of irrep ν in the decomposition of the Kronecker product $\lambda \otimes \mu$. Let $\bar{\lambda}$ denote the representation conjugate to λ .

Theorem 1: *For a given pair (λ, μ) of irreps of the simple Lie algebra \mathfrak{g} , the total multiplicity $\sum_\nu N_{\lambda\mu}^\nu$ satisfies*

$$\sum_\nu N_{\lambda\mu}^\nu = \sum_\nu N_{\bar{\lambda}\mu}^\nu \quad (1.1)$$

Equivalently, since $N_{\lambda\mu}^\nu = N_{\bar{\nu}\bar{\mu}}^{\bar{\lambda}}$,

$$\sum_\lambda N_{\lambda\mu}^\nu = \sum_\lambda N_{\lambda\mu}^{\bar{\nu}} \quad (1.2)$$

Of course the Theorem is non-trivial only in cases where \mathfrak{g} has complex representations, *i.e.* $\mathfrak{g} = A_n$, $D_{n=2s+1}$ or E_6 . Although this looks like a classroom exercise in group theory, we couldn't find either a reference in the literature or a simple and compact argument and we had to resort to a case by case analysis, see Sect 2 below. Note also that this property is not a trivial consequence of the general representation theory of groups; in particular, it does not hold in general in finite groups, see Sect 7 below for counterexamples based on finite subgroups of $SU(3)$.

Theorem 1 is also valid for the *fusion* multiplicities of integrable representations of affine Lie algebras taken at some level k . Such representations are the objects of a fusion category with a finite number of simple objects that will just be called *irreps*, for short. These simple objects (and the category itself) could also be built in terms of irreducible representations of quantum groups at roots of unity that have non-vanishing quantum dimension. One sometimes refers to this framework by saying that we consider the fusion category defined by \mathfrak{g} at level k , but for definiteness, when needed, we shall use the language of affine algebras, and denote $\hat{\mathfrak{g}}_k$ the affine algebra of type \mathfrak{g} at the finite integer level k . Then we have, using the notation $\hat{N}_{\lambda\mu}^\nu$ for the fusion multiplicities, (for completeness a label k should be appended to this notation but will be omitted)

Theorem 2: Eq. (1.1) (or (1.2)) is valid for any pair (λ, μ) of irreps of the fusion category defined by $\hat{\mathfrak{g}}_k$ at level k

$$\sum_{\nu} \hat{N}_{\lambda\mu}^\nu = \sum_{\nu} \hat{N}_{\bar{\lambda}\mu}^\nu \quad (1.3)$$

$$\sum_{\lambda} \hat{N}_{\lambda\mu}^\nu = \sum_{\lambda} \hat{N}_{\lambda\mu}^{\bar{\nu}}. \quad (1.4)$$

Part of the proof given in Sect 2 can be used in that case, but the discussion needs nevertheless to be extended, so that the proof of Theorem 2 is given in Sect 3. Notice that the theorem for simple algebras follows from that for affine algebras, provided the level is chosen large enough.

Now in that context of affine algebras, the multiplicities $\hat{N}_{\lambda\mu}^\nu$ are given by the Verlinde formula [1] in terms of the unitary modular S matrix

$$\hat{N}_{\lambda\mu}^\nu = \sum_{\kappa} \frac{S_{\lambda\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}}, \quad (1.5)$$

where the weight 0 refers to the identity representation. We recall that the matrix S is symmetric $S_{\lambda\kappa} = S_{\kappa\lambda}$ and satisfies the following properties

$$S^\dagger = S^{-1} = S^3 = SC = CS \quad (1.6)$$

where $C = S^2$ is the conjugation matrix: $C_{\lambda\lambda'} = \delta_{\lambda'\bar{\lambda}}$ from which it follows that

$$S_{\bar{\lambda}\kappa} = S_{\lambda\bar{\kappa}} = S_{\lambda\kappa}^*. \quad (1.7)$$

Then we have the (apparently) stronger constraint on $\Sigma(\kappa) := \sum_{\nu} S_{\nu\kappa}$

Theorem 3: $\Sigma(\kappa) := \sum_{\nu} S_{\nu\kappa} = 0$ if $\kappa \neq \bar{\kappa}$.

That Theorem 3 implies Theorem 2 is readily seen:

$$\begin{aligned} \sum_{\nu} \hat{N}_{\lambda\mu}^\nu &= \sum_{\kappa} \frac{S_{\lambda\kappa} S_{\mu\kappa} \sum_{\nu} S_{\nu\kappa}^*}{S_{0\kappa}} \stackrel{(\text{Th.3})}{=} \sum_{\kappa=\bar{\kappa}} \frac{S_{\lambda\kappa} S_{\mu\kappa} \sum_{\nu} S_{\nu\kappa}^*}{S_{0\kappa}} \\ &\stackrel{(1.7)}{=} \sum_{\nu} \sum_{\kappa=\bar{\kappa}} \frac{S_{\bar{\lambda}\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}} = \sum_{\nu} \hat{N}_{\bar{\lambda}\mu}^\nu. \end{aligned} \quad (1.8)$$

As we shall see below, (Sect 4), Theorem 3 also follows from Theorem 2, so that the two statements are in fact equivalent.

Theorem 3 states that $\sum_{\nu} S_{\nu\kappa}$ vanishes if κ is a complex representation. Or equivalently it may be non-zero only if κ is self-conjugate. As is well known this covers two cases, *real* representations and *quaternionic*, also known as *pseudoreal*, representations. We show in Sect 5 that

Theorem 4: *Let κ be an irrep of $\hat{\mathfrak{g}}_k$. If κ is of quaternionic type, the sum $\Sigma(\kappa) = \sum_{\nu} S_{\nu\kappa}$ vanishes.*

The sum $\sum_{\nu} S_{\nu\kappa}$ may thus be non-zero only if κ is a real representation. Actually this sum can sometimes vanish, even for real representations, either because it is forced by some automorphism of the Weyl alcove, or because of some accidental property of the representation κ . We return to this question in Sect 6.

2 Sum of multiplicities (classical case). Proof of Theorem 1

The proof will be done in two steps. We first prove it for λ one of the fundamental representations ω_p , $p = 1, \dots, n$, and μ arbitrary; and then use the fact that any N_{λ} is a polynomial in $N_{\omega_1}, \dots, N_{\omega_n}$.

Lemma 1: *Theorem 1 holds for any fundamental weight $\lambda = \omega_p$.*

We recall a well known method of calculation of the multiplicities $N_{\lambda\mu}^{\nu}$ for two given h.w. λ and μ , often called the Racah–Speiser algorithm [2–4]. Here and below we write the components of weights along the basis of fundamental weights (Dynkin labels). Let ρ stand for the Weyl vector, i.e. the sum of all fundamental weights (or half the sum of positive roots) of \mathfrak{g} . Consider the set of weights $\sigma = \lambda' + \mu + \rho$ where λ' runs over the weight system $[\lambda]$ of the irrep of h.w. λ . Three cases may occur:

- i) if all Dynkin labels of σ are positive, $\lambda' + \mu$ contributes to the sum over h.w. ν with a multiplicity equal to the multiplicity of σ (i.e. of λ');
- ii) if σ or any of its images under the Weyl group has a vanishing Dynkin label, i.e. if σ is on the edge of a Weyl chamber, $\lambda' + \mu$ does not contribute to the sum over ν ;
- iii) if σ has negative (but no vanishing) Dynkin labels, and is not of the type discussed in case (ii), it may be mapped inside the fundamental Weyl chamber by a unique element w of the Weyl group. The weight $w[\sigma] - \rho$ contributes with a multiplicity $\text{sign}(w)$ to the sum over ν .

This is summarized in the formula

$$N_{\lambda\mu}^{\nu} = \sum_{\lambda' \in [\lambda]} \sum_{\substack{w \in W \\ w[\lambda' + \mu + \rho] - \rho \in P_+}} \text{sign}(w) \delta_{\nu, w[\lambda' + \mu + \rho] - \rho} \quad (2.1)$$

in which P_+ is the fundamental Weyl chamber ($\nu \in P_+ \Leftrightarrow \nu_i \geq 0 \ \forall i = 1, \dots, n$).

Remarks. 1. In practice, it may not always be immediately obvious to discover that a shifted weight σ belongs to the edge of a Weyl chamber and therefore trivially contributes to the problem, but one can easily discard at least those σ with one or several Dynkin labels equal to 0 since they obviously

belong to the walls of the fundamental chamber. In any case the trivial σ 's that would not be recognized as such will be mapped, at a later stage, to the walls of the fundamental Weyl chamber, and they can be removed then. Note that in formula (2.1) these cases of type (ii) automatically cancel out, as they contribute with two Weyl elements of opposite signatures.

2. The irreps ν that appear in the decomposition into irreps of the tensor product $\lambda \otimes \mu$ are obtained (together with their multiplicities) from the non-trivial contributions (i) and (iii). The same weight ν can sometimes be obtained both from (i) and (iii), possibly with different signs. Its final multiplicity is the *algebraic* sum of its partial multiplicities.

3. Notice that, as a consequence of the above method, the sum over ν of multiplicities $N_{\lambda\mu}^\nu$ should be smaller than the dimensions of any of the two irreps λ and μ entering the tensor product, $\sum N_{\lambda\mu}^\nu \leq \inf(\dim(\lambda), \dim(\mu))$.

We shall now see that for all the complex fundamental representations of the A , D and E_6 algebras (with one exception in E_6 , see below), we are in case (i) or (ii), and that for $\lambda = \omega_p$ or $\lambda = \bar{\omega}_p$, the occurrences of (ii) are equinumerous, thus proving the Lemma.

For each of the fundamental representations ω_p of the A_n algebra ($p = 1, \dots, n$), for the spinorial representations¹ ω_{n-1} and ω_n of the D_n ($n = 2s + 1$) algebra, and for the 27-dimensional fundamental representations ω_1 and ω_5 of E_6 , the Dynkin labels of the weights λ' of the weight system of $\lambda = \omega_p$ take the value 0 or ± 1 . Thus after addition of ρ (whose Dynkin labels are all equal to 1), the Dynkin labels of $\sigma = \lambda' + \rho + \mu$ are never negative and the case (iii) above never occurs. On the other hand, case (ii) occurs whenever some Dynkin label of μ vanishes while the corresponding one in λ' equals -1 . It is easy to check by inspection that there is the same number of weights with -1 entries at given locations $1 \leq i_1 < i_2 < \dots < i_q \leq n$ in the weight systems of any ω_p and $\bar{\omega}_p$. For a given μ , there is thus an equal number of occurrences of cases of type (ii) for the fundamental weights ω_p and $\bar{\omega}_p$.

To complete the proof of Lemma 1, we still have to consider the case of the complex, 351-dimensional, representations ω_2 and $\bar{\omega}_2 = \omega_4$ of E_6 (notice that ω_2 is also the antisymmetric tensor square of ω_1). This requires a particular analysis because the weight system of ω_2 (or of ω_4) contains weights with Dynkin labels equal to -2 , so that when the corresponding label of μ vanishes, we are in the situation (iii). For the sake of clarity, this detailed discussion is relegated to Appendix A.1.

Lemma 2: *Theorem 1 holds for any product of the fundamental representations.*

In the following it will be convenient to use an alternative notation for the multiplicities $N_{\lambda\mu}^\nu$ and to regard them as the (μ, ν) entry of the matrix N_λ . We have proved in Lemma 1 that for any p

$$\sum_\nu (N_{\omega_p})_\mu^\nu = \sum_\nu (N_{\bar{\omega}_p})_\mu^\nu .$$

¹those are the only fundamental complex representations of the D_{2s+1} case.

This, together with the commutativity of the N matrices, entails that for any monomial $N_{\omega_{j_1}} \cdots N_{\omega_{j_q}}$

$$\begin{aligned}
\sum_{\nu} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_q}} \right)_{\mu}^{\nu} &= \sum_{\nu'} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}} \right)_{\mu}^{\nu'} \sum_{\nu} \left(N_{\omega_{j_q}} \right)_{\nu'}^{\nu} \\
&\stackrel{(\text{Lemma 1})}{=} \sum_{\nu'} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}} \right)_{\mu}^{\nu'} \sum_{\nu} \left(N_{\bar{\omega}_{j_q}} \right)_{\nu'}^{\nu} \\
&= \sum_{\nu} \left(N_{\bar{\omega}_{j_q}} N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}} \right)_{\mu}^{\nu} = \cdots \\
&= \sum_{\nu} \left(N_{\bar{\omega}_{j_1}} \cdots N_{\bar{\omega}_{j_q}} \right)_{\mu}^{\nu} \tag{2.2}
\end{aligned}$$

which exhibits the product of the conjugate fundamental representations.

2) Now it is also well known [4,5] that any irreducible representation may be obtained from a suitable combination of tensor products of the fundamentals. Or in other words, any matrix N_{λ} is some polynomial (with integer coefficients)² of the commuting $N_{\omega_1}, \dots, N_{\omega_n}$: $N_{\lambda} = P_{\lambda}(N_{\omega_1}, \dots, N_{\omega_n})$ and $N_{\bar{\lambda}} = P_{\lambda}(N_{\bar{\omega}_1}, \dots, N_{\bar{\omega}_n})$. Thus the property proved above for any monomial establishes the general statement and completes the proof.

3 Sum of multiplicities (affine/quantum case). Proof of Theorem 2

3.1 Levels and automorphisms

Let P_+^k be the set of integrable weights of the affine algebra $\hat{\mathfrak{g}}$ at level k [6]. Each weight of P_+^k is completely specified by a dominant weight λ of the underlying classical algebra \mathfrak{g} , restricted by the condition $\mathcal{K}(\lambda) \leq k$ where \mathcal{K} is the linear form $\mathcal{K}(\lambda) := \langle \lambda, \theta \rangle$ and θ is the highest root of \mathfrak{g} . We shall call *level* of a weight λ the integer $\mathcal{K}(\lambda)$. Therefore a weight exists in a representation of level k when its level is smaller than or equal to k . By another slight abuse of notation, λ will denote both the weight of $\hat{\mathfrak{g}}$ and the corresponding weight in \mathfrak{g} . We refer to the subset of λ such that $\mathcal{K}(\lambda) = k$ as “the back wall” (of the Weyl alcove P_+^k). It is also convenient to introduce the additional Dynkin label $\lambda_0 = k - \mathcal{K}(\lambda)$ of the affine weight λ : clearly λ_0 vanishes on the back wall.

Each of the algebras $\hat{\mathfrak{g}}$ with complex representations, i.e. \hat{A}_n , \hat{D}_{2s+1} and \hat{E}_6 , has the well known properties

- the set P_+^k of integrable weights at level k is invariant under the action of an automorphism ζ ;
- there exists a \mathbb{Z}_N -grading τ on the weights of P_+^k : $N = n + 1$ for \hat{A}_n , $N = 4$ for \hat{D}_{2s+1} and $N = 3$ for \hat{E}_6 ;
- the modular S -matrix satisfies the relation [7]

$$S_{\zeta(\mu)\kappa} = e^{2\pi i \tau(\kappa)/N} S_{\mu\kappa}. \tag{3.1}$$

²a generalized Chebyshev polynomial [4].

The value of the level $\mathcal{K}(\lambda)$ may be calculated easily from the expansion of the highest root θ in terms of simple roots (Coxeter–Kac labels): $\theta = (1, 1, \dots, 1)$, $(1, 2, 2, \dots, 2, 1, 1)$, $(1, 2, 3, 2, 1, 2)$ for A_n, D_{2s+1}, E_6 respectively. The expressions of $\mathcal{K}(\lambda)$, the automorphisms, the \mathbb{Z}_N -grading and the conjugates in the three above algebras are gathered in Appendix B. One can check on these expressions that the level of a weight is invariant by conjugation: $\mathcal{K}(\mu) = \mathcal{K}(\bar{\mu})$. Moreover the automorphism ζ and the complex conjugation satisfy the consistency relation

$$\zeta(\bar{\mu}) = \overline{\zeta^{-1}(\mu)}, \quad (3.2)$$

and by iteration

$$\zeta^p(\bar{\mu}) = \overline{\zeta^{-p}(\mu)} \quad \forall p. \quad (3.3)$$

For the A_n algebra one finds that $\mathcal{K}(\zeta(\mu)) = k - \mu_n$, $\mathcal{K}(\zeta^{-1}(\mu)) = k - \mu_1$ and more generally

$$\mathcal{K}(\zeta^{-p}(\mu)) = k - \mu_p, \quad (3.4)$$

while for the D_{2s+1} case,

$$\mathcal{K}(\zeta(\mu)) = k - \mu_{2s}, \quad \mathcal{K}(\zeta^{\pm 2}(\mu)) = k - \mu_1, \quad \mathcal{K}(\zeta^{-1}(\mu)) = k - \mu_{2s+1}, \quad (3.5)$$

and for the E_6 case

$$\mathcal{K}(\zeta(\mu)) = k - \mu_5, \quad \langle \mathcal{K}(\zeta^{-1}(\mu)) = k - \mu_1. \quad (3.6)$$

ζ is an automorphism of the fusion rules as a consequence of (1.5) and (3.1)

$$\hat{N}_{\lambda\zeta(\mu)}^{\zeta(\nu)} = \sum_{\kappa} \frac{S_{\lambda\kappa} S_{\zeta(\mu)\kappa} S_{\zeta(\nu)\kappa}^*}{S_{0\kappa}} = \sum_{\kappa} \frac{S_{\lambda\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}} = \hat{N}_{\lambda\mu}^{\nu}. \quad (3.7)$$

This implies that the sum of multiplicities satisfy

$$\sum_{\nu} \hat{N}_{\lambda\mu}^{\nu} = \sum_{\nu} \hat{N}_{\lambda\zeta(\mu)}^{\nu}. \quad (3.8)$$

3.2 Fusion coefficients

There are several alternative routes to determine the fusion coefficients. Let us quote three of them. The first is the Verlinde formula (1.5), which relies on the knowledge of the modular S -matrix.

Secondly one may use an affine generalization of the Racah–Speiser algorithm described in eq. (2.1)

$$\hat{N}_{\lambda\mu}^{\nu} = \sum_{\lambda' \in [\lambda]} \sum_{\substack{w \in \widehat{W} \\ w[\lambda' + \mu + \rho] - \rho \in P_+^k}} \text{sign}(w) \delta_{\nu, w[\lambda' + \mu + \rho] - \rho} \quad (3.9)$$

The modification is twofold : the fundamental Weyl chamber P_+ is replaced by P_+^k , the Weyl alcove P_+^k of level k ; and the sum runs now over elements of the *affine* Weyl group \widehat{W} , of which the reflection s_0 across the shifted back wall is the new generator. What is referred to as the shifted back wall is the hyperplane of equation $\mathcal{K}(\lambda) = k + h^\vee$, and the reflection s_0 acts according to $s_0[\lambda] = \lambda + (k + h^\vee - \mathcal{K}(\lambda)) \frac{2\theta}{\langle \theta, \theta \rangle}$, where $h^\vee = 1 + \langle \rho, \theta \rangle$ is the dual Coxeter number. Just like in Sect 2, weights λ' which are such that $\lambda' + \mu + \rho$ lies either on an ordinary wall of the Weyl chamber, or on the shifted back wall, or on one of their images by \widehat{W} , do not contribute to the sum.

Thirdly, the fusion coefficients $\hat{N}_{\lambda\mu}^\nu$ and the ordinary multiplicities $N_{\lambda\mu}^\nu$ occurring in the “horizontal” algebra \mathfrak{g} are related by the Kac–Walton formula [8],

$$\forall \lambda, \mu, \nu \in P_+^k \quad \hat{N}_{\lambda\mu}^\nu = \sum_{\substack{w \in \bar{W} \\ w[\nu+\rho]-\rho \in P_+}} \text{sign}(w) N_{\lambda\mu}^{w[\nu+\rho]-\rho}. \quad (3.10)$$

As far as the proof of Theorem 2 is concerned, the first method (Verlinde formula) does not seem appropriate, unless some additional properties of that matrix (in fact our Theorem 3) are proved beforehand. On the other hand, repeating the method of Sect 2 with the affine version of the Racah–Speiser algorithm leads in a straightforward way to a proof, as we shall see in the next subsection. Using the results of Sect 2 on sums of tensor product multiplicities together with (3.10) and the automorphism ζ of section 3.1 is another tantalizing possibility, which however seems to be applicable only to a subset of cases. We return to this point at the end of next subsection.

3.3 Proof of Theorem 2

As in Sect 2, we take λ to be the highest weight of one of the complex fundamentals of the affine algebra $\hat{\mathfrak{g}}$ with $\mathfrak{g} = A_n$, $D_{n=2s+1}$ or E_6 . Again, in the latter case, we treat separately the weights ω_2 and ω_4 (their level is 2). Each of the other cases ($\lambda = \omega_p$, $p = 1, \dots, n$, in A_n , ω_{2s} or ω_{2s+1} in $D_{n=2s+1}$, and ω_1 or ω_5 in E_6) has a level $\mathcal{K}(\omega_p) = 1$, and all the weights λ' of the representation λ have a level $\mathcal{K}(\lambda') = \pm 1$ or 0, as is readily checked on their expression.

We then follow the same steps as in Sect 2: for any weight $\mu \in P_+^k$, hence with all its Dynkin labels (including the affine label μ_0) non-negative, and for any $\lambda' \in [\lambda = \omega_p]$, one sees that $\sigma = \lambda' + \mu + \rho$ has non-negative Dynkin labels σ_i , $i = 1, \dots, n$ and likewise

$$\sigma_0 = k + h^\vee - \mathcal{K}(\sigma) = (k - \mathcal{K}(\mu)) + (1 - \mathcal{K}(\lambda')) \geq 0. \quad (3.11)$$

Hence no non-trivial w has to be applied to σ to bring it back (after subtraction of ρ) to P_+^k . But some of these σ may lie on a wall and will not contribute to the sum in (3.9), and this occurs whenever one or several of the Dynkin labels μ_i , $i = 0, \dots, n$ vanish. In view of the discussion of Sect 2 for the finite case, it suffices to examine the situation when σ lies on the shifted back wall, i.e. σ_0 vanishes, and (3.11) says this occurs whenever μ lies on the back wall of P_+^k and $\mathcal{K}(\lambda') = +1$. Since for any λ' of level 1, its conjugate $\bar{\lambda}'$ has also level 1, the number of these occurrences is the same for $\lambda = \omega_p$ and $\bar{\omega}_p$, and like in the finite case of Sect 2, this implies the equality $\sum_\nu \hat{N}_{\omega_p\mu}^\nu = \sum_\nu \hat{N}_{\bar{\omega}_p\mu}^\nu$. The case of $\lambda = \omega_2$ or $= \omega_4$ for E_6 has again to be treated separately and will be relegated to Appendix A.2.

Once it has been established for λ one of the fundamentals, Theorem 2 then follows in general from the fact that the fusion ring is polynomially generated by the fundamental fusion matrices \hat{N}_{ω_p} [4].

An alternative route using the Kac–Walton formula (3.10) is also applicable to the A_n case (and also to D_{2s+1} case *at odd level k*). The method stems from the observation that when λ or μ are sufficiently off the back wall, so that all ν such that $N_{\lambda\mu}^\nu \neq 0$ are themselves in P_+^k , only $w = 1$ contributes to the sum in (3.10) and $\hat{N}_{\lambda\mu}^\nu$ does not differ from $N_{\lambda\mu}^\nu$. Unfortunately the method does not seem to be of general validity and we have thus to rely on the more systematic proof given previously.

4 Proof of Theorem 3

We want to show (Th 3) that if $\kappa \neq \bar{\kappa}$, then $\Sigma(\kappa) = \sum_{\nu} S_{\nu\kappa} = 0$.

If $\kappa \neq \bar{\kappa}$, there are two cases, either $\tau(\kappa)$ vanishes, or it does not. The proof splits then naturally into two parts.

First observe that for any κ of non-vanishing τ , $\sum_{\lambda} S_{\lambda\kappa} = 0$. Indeed

$$\sum_{\lambda} S_{\lambda\kappa} = \sum_{\lambda} S_{\zeta(\lambda)\kappa} = e^{2\pi i \tau(\kappa)/N} \sum_{\lambda} S_{\lambda\kappa}. \quad (4.1)$$

As we shall now see, if κ is such that $\sum_{\lambda} S_{\lambda\kappa} \neq 0$, then for any μ , we have $S_{\mu\kappa} = S_{\mu\bar{\kappa}}$, and for $\kappa \neq \bar{\kappa}$, this leads to a contradiction. Therefore, if κ is such that $\sum_{\lambda} S_{\lambda\kappa} \neq 0$, then $\kappa = \bar{\kappa}$. Equivalently, if $\kappa \neq \bar{\kappa}$, then $\sum_{\lambda} S_{\lambda\kappa} = 0$, even if $\tau(\kappa)$ vanishes.

Completing the proof therefore requires two small lemmas that we now discuss in detail.

Verlinde formula (1.5) implies

$$S_{\lambda\kappa} S_{\mu\kappa} = \sum_{\nu} \hat{N}_{\lambda\mu}^{\nu} S_{\nu\kappa} S_{0\kappa}. \quad (4.2)$$

and we have proved that $\sum_{\lambda} \hat{N}_{\lambda\mu}^{\nu} = \sum_{\lambda} \hat{N}_{\lambda\mu}^{\bar{\nu}}$, see (1.4). Therefore, for any κ ,

$$\begin{aligned} (\sum_{\lambda} S_{\lambda\kappa}) S_{\mu\kappa} &= \sum_{\nu} (\sum_{\lambda} \hat{N}_{\lambda\mu}^{\nu}) S_{\nu\kappa} S_{0\kappa} = \sum_{\nu} (\sum_{\lambda} \hat{N}_{\lambda\mu}^{\bar{\nu}}) S_{\nu\kappa} S_{0\kappa} \\ &= \sum_{\nu} (\sum_{\lambda} \hat{N}_{\lambda\mu}^{\nu}) S_{\bar{\nu}\kappa} S_{0\kappa} = \sum_{\nu} (\sum_{\lambda} \hat{N}_{\lambda\mu}^{\nu}) S_{\nu\bar{\kappa}} S_{0\kappa} \\ &= \sum_{\nu} (\sum_{\lambda} \hat{N}_{\lambda\mu}^{\nu}) S_{\nu\bar{\kappa}} S_{0\bar{\kappa}} = \sum_{\lambda} S_{\lambda\bar{\kappa}} S_{\mu\bar{\kappa}} = \sum_{\lambda} S_{\lambda\kappa} S_{\mu\bar{\kappa}} = (\sum_{\lambda} S_{\lambda\kappa}) S_{\mu\bar{\kappa}} \end{aligned} \quad (4.3)$$

where we used the fact that $S_{0\bar{\kappa}} = S_{0\kappa}$ is real (it is a quantum dimension up to a real factor S_{00}), and that summations over ν or $\bar{\nu}$ are equivalent. Therefore we have proved

Lemma 3: *For any κ such that $\sum_{\lambda} S_{\lambda\kappa} \neq 0$ (hence of vanishing τ), and for any μ , we have*

$$S_{\mu\kappa} = S_{\mu\bar{\kappa}}. \quad (4.4)$$

To complete the proof, we have to show that this situation cannot occur for κ complex.

Lemma 4: *For any complex κ , i.e. $\kappa \neq \bar{\kappa}$, there exists a weight $\mu \in P_+^k$ such that*

$$S_{\mu\kappa} \neq (S_{\mu\kappa})^* = S_{\mu\bar{\kappa}}. \quad (4.5)$$

Note this holds irrespectively of whether $\tau(\kappa)$ vanishes or not.

Proof. For such a $\kappa \neq \bar{\kappa}$, (the h.w. of a complex representation), the fusion matrices \hat{N}_{κ} and $\hat{N}_{\bar{\kappa}}$ are different, since $(\hat{N}_{\kappa})_0^{\kappa} = 1$ whereas $(\hat{N}_{\bar{\kappa}})_0^{\kappa} = 0$. But these two matrices are diagonalized in the same basis through Verlinde's formula, with eigenvalues $S_{\kappa\mu}/S_{0\mu}$, resp. $S_{\bar{\kappa}\mu}/S_{0\mu}$. Thus there is at least one distinct pair of eigenvalues $S_{\kappa\mu} \neq S_{\bar{\kappa}\mu}$. The lemma is proved.

Lemma 4, together with Lemma 3 (4.4), implies that $\sum_{\lambda} S_{\lambda\kappa} \neq 0$ is only possible if $\kappa = \bar{\kappa}$, and this completes the proof of Theorem 3.

Comment

The previous discussion was needed to handle the general case where the representation κ is complex, but let us remember that for those particular complex representations of non-vanishing τ , the proof of the vanishing of $\sum_{\lambda} S_{\lambda\kappa}$ is immediate. In the case of A_n , such a simplified proof can be given for instance if κ is a fundamental representation, and more generally when $\sum_j j \kappa_j \neq 0 \pmod{n+1}$. In the case of E_6 , assuming κ complex, ie $\kappa_1 \neq \kappa_5$ or $\kappa_2 \neq \kappa_4$, such a simplified proof can also be given for the complex fundamentals (100000), (010000) and their conjugates (000010), (000100), and more generally when $2\kappa_1 + \kappa_2 + 2\kappa_4 + \kappa_5 = 1, 2 \pmod{3}$.

5 The case of quaternionic representations

5.1 The case of $\mathfrak{su}(2)$

For the $\widehat{\mathfrak{su}}(2)_k$ algebra, the integrable weights are $\lambda \in \{0, 1, \dots, k\}$. Denote $h = k + 2$ for brevity. Then $S_{\lambda\kappa} = \sqrt{\frac{2}{h}} \sin \frac{(\lambda+1)(\kappa+1)\pi}{h}$ and

$$\sqrt{\frac{h}{2}} \sum_{\lambda=0}^k S_{\lambda\kappa} = -\frac{\cos \frac{\pi(\kappa+1)(2h-1)}{2h} - \cos \frac{\pi(\kappa+1)}{2h}}{2 \sin \frac{\pi(\kappa+1)}{2h}} = \frac{(1 - (-1)^{\kappa+1}) \cos \frac{\pi(\kappa+1)}{2h}}{2 \sin \frac{\pi(\kappa+1)}{2h}}$$

which vanishes for κ odd, corresponding to quaternionic (half-integer spin) representations. This result, obtained here in an explicit manner, will be recovered and generalized below for all integrable weights corresponding to irreducible representations κ of quaternionic type.

5.2 A case by case study

In all cases we shall compare the results of appendix C describing representation types for irreducible representations with the results gathered in appendix B, that allow us to calculate the values of the \mathbb{Z}_N grading $\tau = \tau(\mu)$ for quaternionic representations. We shall see that for all simple Lie groups, and for quaternionic representations the quantity τ (or at least one of the possible τ 's associated with an appropriate automorphism) does not vanish. Like in section 4 we then consider the $S_{\lambda\kappa}$ matrix elements and notice that the exponential factor appearing in (3.1) or in (4.1) is not equal to 1 for such representations. This shows immediately that $\sum_{\lambda} S_{\lambda\kappa} = 0$ if κ is of quaternionic type.

5.2.1 The case $A_n \sim \mathfrak{su}(n+1)$

Quaternionic representations may only exist when $n+1 = 2 \pmod{4}$. Their highest weight μ should have Dynkin labels that are symmetric with respect to the middle point (the position labeled $(n+1)/2$), and the middle Dynkin label should be odd. Calculating the N -ality τ of these representations (here $N = n+1$) we see immediately that only the middle term $(n+1)/2 \mu_{(n+1)/2}$ survives: being a product of two odd factors, it is also odd and does not vanish modulo the even integer $n+1$.

5.2.2 The case $B_n \sim \mathfrak{so}(2n+1)$

Irreps of B_n are quaternionic if and only if, simultaneously, $n = 1$ or 2 modulo 4 and μ_n is odd. Notice that among *fundamental* irreps, only the last one (the spinorial) *can be* quaternionic. This result may be put in relation with Clifford algebra considerations since, in terms of spin groups $\text{Spin}(d)$ with d odd, quaternionic representations appear when d is equal to 3 or 5 modulo 8 . The \mathbb{Z}_2 grading τ (a “2-ality” in this case) of a quaternionic irrep never vanishes since μ_n is odd for such representations.

5.2.3 The case $C_n \sim \mathfrak{sp}(2n)$

Convention: the last root (to the right) is long. Irreps are of quaternionic type whenever $\mu_1 + \mu_3 + \mu_5 + \dots + \mu_m$ is odd (where $m = n$ if n is odd and $m = n - 1$ if n is even). But then, their \mathbb{Z}_2 grading τ is equal to 1 , and the discussion goes as before with the same conclusion.

5.2.4 The case $D_n \sim \mathfrak{so}(2n)$

Convention: the end points of the “fork” of the Dynkin diagram are to the right, in positions $n - 1$ and n . We assume $n \geq 3$. Remember that $D_3 \sim A_3$. The irreps are quaternionic if and only if, simultaneously, $n = 2 \pmod{4}$ and $\mu_{n-1} + \mu_n$ is odd. This implies that either μ_{n-1} is odd or μ_n is odd, but not both.

It is not too difficult to prove that, in such a case, one of the two gradings τ' or τ'' associated with the two generators ζ' and ζ'' of $\mathbb{Z}_2 \times \mathbb{Z}_2$ will not vanish, but it is much simpler, and actually immediate, to use the product of these two generators (see the table in appendix B), with associated grading τ''' since it gives directly $\tau'''(\mu) = 2(\mu_{n-1} + \mu_n) \pmod{4}$, so that $\tau'''(\mu) = 2 \neq 0$ for quaternionic representations.

5.2.5 The case E_7

An irrep μ is of quaternionic type iff $\mu_1 + \mu_3 + \mu_7$ is odd (read our convention for vertices of E_7 at the end of appendix B). The center is now \mathbb{Z}_2 and the associated grading is $\tau(\mu) = \mu_1 + \mu_3 + \mu_7 \pmod{2}$. We reach immediately the conclusion that τ does not vanish for irreps of quaternionic type.

5.2.6 The cases $D_{\text{odd}}, G_2, F_4, E_6, E_8$

All the irreps of G_2, F_4, E_8 are self-conjugate of real type. Not all the irreps of E_6 are self-conjugate, but all self-conjugate irreps are of real type. The irreps of D_{odd} are real or complex according as the last two components of their highest weight, but they are never quaternionic.

Therefore, in the above cases, there is nothing else to discuss, as far as quaternionic irreps are concerned.

This case by case study completes the proof of Theorem 4.

6 The case of real representations

It may happen that $\Sigma(\kappa) = \sum_{\lambda} S_{\lambda\kappa}$ still vanishes for some representation μ of real type. This can be the consequence of the existence of some non-trivial automorphism of the Weyl alcove associated with a non-zero grading τ , but it can just be an accidental property of the chosen representation. Notice that there are no non-trivial automorphisms for F_4 , G_2 and E_8 anyway.

6.1 About the vanishing of $\Sigma(\kappa)$, for κ real, implied by automorphisms with non-zero associated grading

Using together the tables of appendices B and C, it is easy to see that, for real representations, τ is always 0 for A_n , C_n , E_6 and E_7 . Hence, in these cases, there is no constraint on representations μ of real type coming from the existence of automorphisms, and we therefore expect that $\Sigma(\kappa)$ will be generically non-vanishing.

For irreps of real type of B_n and D_n we find non-trivial constraints.

B_n . If $n = 0, 3 \pmod{4}$, then choosing the last component κ_n of κ to be odd, leads to a non-trivial τ , so that the sum $\Sigma(\kappa)$ vanishes. If $n = 1, 2 \pmod{4}$ we do not find any constraint on this sum for real representations (they are such that κ_n is even), but remember that this sum vanishes when κ_n is odd since the representation is then quaternionic.

D_n (here n can be even or odd). Take κ an irrep of real type (see table in appendix C), then the sum $\Sigma(\kappa)$ is zero as soon as one of the following three quantities $2 \sum_{j=1, j \text{ odd}}^{n-3} \kappa_j + 2\kappa_n$, $2 \sum_{j=1, j \text{ odd}}^{n-3} \kappa_j + 2\kappa_{n-1}$, or $2\kappa_{n-1} + 2\kappa_n$ does *not* vanish modulo 4.

6.2 About accidental vanishing of $\Sigma(\kappa)$, for κ real

Notice first that vanishing properties of $\Sigma(\kappa)$ discussed so far are level independent, in the sense that they will hold for all values of the level k , provided κ itself exists at the chosen level (i.e. $\mathcal{K}(\kappa) \leq k$). This is not so for the accidental vanishing cases that we discuss now. For definiteness let us call “accidental vanishing at level k ” a case where $\Sigma(\kappa) = 0$ although this is not implied by any of the already known criteria, in particular κ should be of real type and the vanishing property should not be the consequence of the existence of already discussed non-trivial automorphisms. The very nature of the problem implies that the best we can do in this section is to mention our numerical observations. Such experiments rest on the calculation of the modular S matrix, for various choices of the Lie algebra \mathfrak{g} , and for relatively small values of the level.

The only accidental vanishing properties that we observed occur in the cases F_4 (we made tests up to level 4) and G_2 (we made tests up to level 12). We know that all representations of these algebras are of real type, and that their Dynkin diagrams do not have automorphisms. In both cases, we noticed nevertheless several cancellations of $\Sigma(\kappa)$ (only for even levels in the case of G_2).

For G_2 , we found 2 cases at level 4, 2 cases at level 6, 5 cases at level 8, 6 cases at level 10, 11 cases at level 12. For F_4 , we found 2 cases at level 3 and 1 case at level 4. These cancellations are level specific but some of them have a tendency, in some sense, to stabilize: indeed some representations κ make Σ vanish at some level but not at higher levels, whereas other κ , that appear at some level and make Σ vanish, seem to stay at higher level (shifted by +2 in the case of G_2). Admittedly we have no explanation at the moment for these observations.

This level dependence of accidental vanishing cases should be contrasted with, for example, a “simple” case like E_6 (that we tested up to level 4) where no accidental vanishing appears. Here, at level 3, one finds 16 weights that make Σ vanish (among the 20 integrable ones), but those 16 are still present among the 34 that make Σ vanish at level 4 (there are 42 integrable representations at that level). As it was shown in previous sections these cancellations are associated with the existence of complex irreps.

6.3 Remark

The type (complex, real or quaternionic) of irreps in the affine/quantum case $\hat{\mathfrak{g}}_k$ at level k is the same as the type obtained classically (ie $k \rightarrow \infty$), for irreps of the associated Lie algebra \mathfrak{g} . The corresponding conditions on Dynkin labels can be found in articles or books on representation theory of Lie groups [9], [10]. One can however take advantage of the finiteness of the number of simple objects in the category defined by \mathfrak{g} at level k to obtain a closed formula generalizing, to this context, the Frobenius-Schur indicator used in the theory of finite groups. Such a formula, that we recall in Appendix C.2 was proposed in [11], see also [12], although we find more handy to use another expression (also given in Appendix C.2). One can, for any chosen example, use this indicator to determine the representation type directly in terms of the S and T matrices, without relying on the classification of representation types for Lie algebras given in appendix C.

7 The case of finite groups

Is there an analogue of Theorem 1 true for finite groups? Let G be a finite group. We label its irreps V_i by an index $i = 1, 2, \dots, r$ and its conjugacy classes C_a by $a = 1, 2, \dots, r$; \bar{i} refers to the complex conjugate irrep of i . Let N_{ij}^k stand for the multiplicity of irrep k in $i \otimes j$. Do we have like in Theorem 1

$$\sum_k N_{ij}^k \stackrel{?}{=} \sum_k N_{\bar{i}\bar{j}}^k. \quad (7.1)$$

We first observe that (7.1) is trivially true for the group \mathbb{Z}_n for which the j -th representation is $z \mapsto z^j$, z a n -th root of 1, $N_{ij}^k = \delta_{i+j, k \bmod n}$ and hence $\sum_k N_{ij}^k = 1 = \sum_k N_{\bar{i}\bar{j}}^k$.

To probe (7.1), we have to consider less trivial groups possessing complex representations and it is natural to look at subgroups of $SU(3)$. Consider for example the subgroup of $SU(3)$ of order 1080, called L or $\Sigma(3 \times 360)$ in the nomenclatures³ of Yau-Yu [13] and of Fairbairn et al [14]. It has 17 conjugacy classes and 17 irreps, including one of dimension 3, that we denote f , which is the restriction of the defining representation of $SU(3)$. On Fig. 1, we display the tensor product graph N_f , computed using the character table given in [15] (see also [16]): its vertices i label the 17

³ Warning: groups $\Sigma(n)$ associated with groups $\Sigma(3 \times n)$ are subgroups of $SU(3)/\mathbb{Z}_3$, not of $SU(3)$.

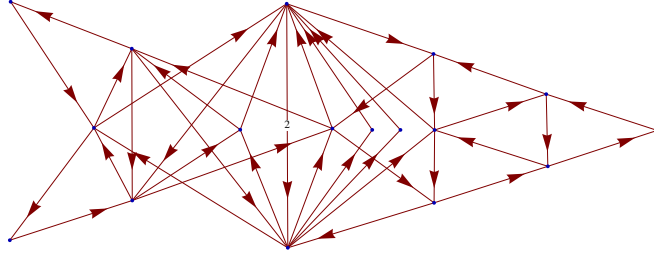


Figure 1: The tensor product graph N_f for the subgroup $\Sigma(1080)$

irreps V_i , and there are N_{fj}^k edges from j to k . A 2 has been appended to the only (vertical) edge for which $N_{fj}^k = 2$, all the others being equal to 1. The graph has been drawn in such a way that complex conjugate representations are images in a reflection through the horizontal axis. Then Theorem 1, if true in that case, would imply that the total number $\sum_k N_{fj}^k$ of outgoing edges from any vertex j equals that from vertex \bar{j} ; or alternatively, that for an arbitrary vertex k , the number $\sum_j N_{fj}^k$ of incoming oriented edges is equal to the number $\sum_j N_{fk}^j$ of outgoing oriented edges.

It is clear on the Figure that this is not true in general, see for example the two vertices in the upper and lower middle positions.

On the other hand, we found that (7.1) holds true for most subgroups of $SU(3)$ but fails for some subgroups like $F = \Sigma(3 \times 72)$ or $L = \Sigma(3 \times 360)$. We could not find the criterion of validity.

As the multiplicity N_{ij}^k may be written as a sum over classes of characters

$$N_{ij}^k = \sum_a \frac{|C_a|}{|G|} \chi_i(a) \chi_j(a) \chi_k^*(a), \quad (7.2)$$

whose analogy with (1.5) is manifest, it is natural to wonder if Theorem 3 admits itself an analogue, whenever (7.1) holds true. In other words, do we have

$$\sum_k \chi_k(a) \stackrel{?}{=} 0 \text{ if } a \neq \bar{a} \quad (7.3)$$

where \bar{a} labels the class of the conjugates⁴ of the elements of C_a . Just like in Sect 1, it is clear that (7.3) implies (7.1), since $\chi_{\bar{i}}(a) = \chi_i(\bar{a})$. And conversely, just like in Sect 4, we can prove that (7.3) follows from (7.1). Thus (7.3) fails for some of the subgroups of $SU(3)$, like $F = \Sigma(3 \times 72)$ or $L = \Sigma(3 \times 360)$.

We conclude that the validity for finite groups of (the analogues of) Theorems 1 and 3 is not to be taken for granted in general.

Its validity for Lie groups and affine algebras might be an indication that the existence of the Weyl group is an important ingredient, but this point should be clarified.

⁴ Here G denotes a concrete subgroup of $SU(3)$, and complex conjugation is well defined.

8 Applications and discussion

8.1 Nimreps and boundaries

The property of the fusion algebra encapsulated in Theorems 2 and 3 has consequences on representations of that algebra. Particularly interesting are the non-negative integer valued matrix representations⁵ (“nimreps”) of the fusion algebra, namely matrices n_λ with non-negative entries $(n_\lambda)_a^b$ satisfying

$$n_\lambda n_\mu = \hat{N}_{\lambda\mu}^\nu n_\nu. \quad (8.1)$$

They describe the action $\lambda a = \sum_b (n_\lambda)_a^b b$ of the fusion ring on its modules and they are known to play a role in various physical or mathematical contexts. In particular in boundary conformal field theory, $(n_\lambda)_a^b$ gives the multiplicity of representation λ for the WZW theory associated with the affine algebra $\hat{\mathfrak{g}}$, on an annulus with boundary conditions labelled by a and b [17, 18]. The nimreps, also known as annular matrices (see for instance [19]), are used, as well, in the context of topological field theories.

In general, these commuting normal matrices may be diagonalized in a common orthonormalized basis ψ in the form

$$n_{\lambda a}^b = \sum_{\kappa \in \mathcal{E}} \psi_a^{(\kappa)} \psi_b^{(\kappa)*} \frac{S_{\lambda\kappa}}{S_{0\kappa}} \quad (8.2)$$

with eigenvalues $\frac{S_{\lambda\kappa}}{S_{0\kappa}}$ of the same form as those of \hat{N}_λ , but labelled by a subset \mathcal{E} of h.w. κ called exponents. The ψ 's enjoy conjugacy properties similar to those of the S matrix, in particular

$$\psi_b^{(\kappa)*} = \psi_b^{(\bar{\kappa})}. \quad (8.3)$$

The subset of exponents is closed under conjugacy, so that the above equation implies immediately $(n_{\bar{\lambda}})_a^b = (n_\lambda)_b^a$ i.e. $n_{\bar{\lambda}} = n_\lambda^T$. The matrices n_λ may be regarded as adjacency matrices of a collection of graphs, with vertices labelled by indices a, b, \dots referring to a particular basis $Vert$ of the chosen module.

Automorphisms ζ of the underlying affine Lie algebra at level k act both on the fusion ring and on its associated modules. They are often called *symmetries*. For instance, the transformation $\lambda \mapsto k - \lambda$ is a symmetry of the fusion ring of $SU(2)$ at level k . It is enough to know the action of the generator(s) described in Appendix B. On the fusion ring, we have $\zeta(\lambda\mu) = \zeta(\lambda)\mu = \zeta(\mu)\lambda$, in particular $\zeta(\lambda) = \zeta(\mathbf{1})\lambda$ where $\mathbf{1} = (0, 0, \dots, 0)$ labels the trivial representation of the Lie algebra. In terms of fusion matrices the symmetry property reads $\hat{N}_{\lambda\zeta(\mu)}^{\zeta(\nu)} = \hat{N}_{\lambda\mu}^\nu$. On a module, the action is specified by setting $\zeta(a) = \zeta(\mathbf{1})a$ for all $a \in Vert$. One obtains immediately $\zeta(\lambda a) = \zeta(\mathbf{1})\lambda a = \lambda\zeta(\mathbf{1})a = \lambda\zeta(a)$. We denote by the same symbol P the matrices describing multiplication by $\zeta(\mathbf{1})$ both in the fusion ring and in the module, i.e. $P = N_{\zeta(\mathbf{1})}$ or $P = n_{\zeta(\mathbf{1})}$. Obviously $\hat{N}_{\zeta(\lambda)} = \hat{N}_\lambda P$ and $n_{\zeta(\lambda)} = n_\lambda P$. Denoting by the same symbol X the two matrices⁶ $\sum_\lambda \hat{N}_\lambda$ and $\sum_\lambda \hat{n}_\lambda$, one obtains immediately $XP = X$ since the action of ζ is one-to-one.

A complex *conjugation* in the module is an involution⁷ $a \mapsto \bar{a}$ such that $\overline{\bar{\lambda}a} = \bar{\lambda}\bar{a}$. If the basis

⁵It may happen that some nimreps, dubbed “non-physical”, do not describe any boundary cft, or in a categorical language, any “module-category” for the chosen fusion category. Unless otherwise specified, we are only interested in the physical ones.

⁶These “path matrices” X are discussed in section 8.4

⁷When conjugation in the fusion ring itself is trivial, there is no need to introduce this concept.

$Vert$ used to label the nimreps is stable as a set under transformations $a \mapsto \zeta(a)$ and $a \mapsto \bar{a}$, the previous conditions read respectively $n_{\lambda\zeta(a)}^{\zeta(b)} = n_{\lambda a}^b$ and $n_{\lambda\bar{a}}^{\bar{b}} = n_{\lambda a}^b$ for all λ, a, b . One can always define a matrix C with $C^2 = 1$ such that $n_{\bar{\lambda}} = C n_{\lambda} C$. From a given conjugation in a module one can obtain another one by composing it with a symmetry. Usually an involution $a \mapsto \bar{a}$ is determined, up to symmetry, from the known conjugacy properties of the set of exponents, but there may nevertheless remain an ambiguity when some exponents have multiplicity higher than 1. The ambiguity in the definition of C reflects a potential ambiguity in the definition of the diagonalizing ψ matrix because C can be defined as $\psi^T \psi$. Notice that the matrix $\psi \psi^T$ gives the restriction of the known conjugation matrix of the Lie algebra at level k to the corresponding set of exponents.

This discussion applies in particular to the nimreps of the $\widehat{su}(2)$ algebra, which are in one-to-one correspondence with the ADE Dynkin diagrams (plus the “tadpole” diagrams⁸ $T_n = A_{2n}/\mathbb{Z}_2$). All irreps at level k are self-conjugate but there is a non-trivial involution P on the A_n diagrams, that induces a non-trivial involution on the $D_{n=2s+1}$ and E_6 diagrams. Here $a \mapsto \zeta(a)$ is just the \mathbb{Z}_2 symmetry of the Dynkin diagram. For the D_{even} diagrams, the matrix P is trivial, although we still have a non-trivial geometrical symmetry that exchanges the two branches of the fork, *i.e.* a graph automorphism⁹. Notice that symmetries of a module structure over the fusion ring, as defined in the text, give rise to automorphisms of fusion graphs, but there may be more of the latter. In the case of nimreps of the $\widehat{su}(3)$ algebra, the various diagrams exhibit several interesting geometrical symmetries, but besides the diagrams of type \mathcal{A} themselves, only the exceptional diagram with self-fusion at level 5 and the diagrams of the conjugated Dstar family, when the level is not 0 modulo 3, admit a non-trivial matrix P inherited from the \mathbb{Z}_3 symmetry of the corresponding fusion algebra. For all these cases $X P = X$ and P is non-trivial.

In the case of the $SU(2)$ WZW model, the equation $X = X P$ means that the total number of representations, *i.e.* of primary fields contributing to the annulus partition function in the presence of boundary conditions a and b is the same as with b.c. a and $\zeta(b)$. This extends to the minimal $c < 1$ cft’s, that are constructed as cosets of the $su(2)$ theories. They are classified by a pair $(A_{h'-1}, G)$ of Dynkin diagrams, G of ADE type and of Coxeter number h . Their boundary conditions are classified by pairs (ρ, a) with $\rho = 1, \dots, h' - 1$ and a a vertex of G [17]. Take one of the cases $G = A, D_{odd}, E_6$. The multiplicity $n_{rs; (\rho, a)}^{(\rho', b)}$ of the (r, s) primary field in the annulus partition function with boundary conditions (ρ, a) on one side and (ρ', b) on the other is not invariant under the symmetry $b \mapsto \zeta(b)$, but the total multiplicity $\sum_s n_{rs; (\rho, a)}^{(\rho', b)}$ is, as one may check for example on the explicit formulae of [20] in the case of E_6 .

In general, conjugacy properties of the ψ ’s imply (or are implied by) conjugacy properties of the n ’s, but Theorem 4 implies stronger properties for *sums* of the n ’s. Following steps similar to those in (1.8) in Sect 1 and making use of $\psi_b^{(\kappa)} = \psi_b^{(\kappa)}$ for real κ , one finds $X = X C$. Indeed,

$$\sum_{\lambda} n_{\lambda} = \left(\sum_{\lambda} n_{\lambda} \right)^T = C \sum_{\lambda} n_{\lambda} = \sum_{\lambda} n_{\lambda} C \iff \sum_{\lambda} n_{\lambda a}^b = \sum_{\lambda} n_{\lambda a}^{\bar{b}} \quad (8.4)$$

Like for S itself, we have observed, in many cases, intriguing sum rules concerning the matrix $\psi = (\psi_a^{(\kappa)})$, involving summations either over the exponents, or over the label a . We hope to return to this analysis in a later work.

⁸that actually describe non-physical nimreps.

⁹Graph automorphisms are permutations π on vertices of a graph such that for all pairs of vertices, $(\pi(a), \pi(b))$ is an edge iff (a, b) is an edge.

8.2 Integrable S-matrices

The nimreps of the previous section have appeared in a different context, that of S-matrices of integrable 2-d field theories. In the study of affine Toda theories or of other integrable 2-d theories based on a simply laced algebra, Braden, Corrigan, Dorey and Sasaki [21] were led to expressions, proved or conjectured, of their scattering \mathcal{S} -matrix. Typically the particles of those theories are in one-to-one correspondence with the vertices of ADE-Dynkin diagrams.

The \mathcal{S}_{ab} matrix describing the scattering of particles a and b is a function of the relative rapidity $\theta = \theta_a - \theta_b$ and satisfies the constraints of

- unitarity $\mathcal{S}_{ab}(\theta)\mathcal{S}_{ab}(-\theta) = I$ and
- crossing $\mathcal{S}_{ab}(\theta) = \mathcal{S}_{b\zeta(a)}(i\pi - \theta)$

which imply that \mathcal{S}_{ab} is $2\pi i$ periodic. Its analytic structure may be investigated in the strip $0 \leq \Im \theta < \pi$, from which the whole period may be recovered. One finds that in that strip, it has poles at $\theta = \vartheta_\ell := \ell \frac{i\pi}{h}$, with h the Coxeter number of the ADE diagram and $\ell = 1, \dots, h-1$. Quite amazingly [22, 23], the multiplicity of the pole at ϑ_ℓ turns out to be $n_{\ell-2a}^b + n_{\ell a}^b$, where by convention $n_{-1} = n_{h-1} = 0$.

In that context, the identity (8.4), rewritten here as $\sum_\lambda n_{\lambda a}^b = \sum_\lambda n_{\lambda \zeta(b)}^a$ because of the symmetry of the n matrices in this case, expresses that the total number of poles of \mathcal{S}_{ab} and of $\mathcal{S}_{\zeta(b)a}$ are equal, in accordance with the crossing relation above¹⁰.

8.3 Sum rules for character polynomials

Call $\chi(\lambda) = \chi(\lambda; t_1, t_2, \dots, t_n)$ the classical character polynomial of the Lie group G , associated with an irreducible representation defined by its highest weight λ . It encodes the weight system of λ : each weight $\ell \in [\lambda]$ occurring with multiplicity a in this weight system gives a Laurent monomial $a t_1^{\ell_1} t_2^{\ell_2} \dots t_n^{\ell_n}$ in $\chi(\lambda)$. Here $(\ell_1, \ell_2, \dots, \ell_n)$ are the Dynkin labels (that can be positive or negative or zero) of ℓ . Evaluation at level k of such a monomial on a weight μ is, by definition $a \exp[2i\pi/(h^\vee + k) \langle \ell_1 \omega_1 + \ell_2 \omega_2 + \dots + \ell_n \omega_n, \mu \rangle]$, where ω_i are the fundamental weights, and it is extended to arbitrary Laurent polynomials by linearity. The obtained value is denoted $\chi(\lambda)[\mu]$. Assuming that λ and μ are two irreducible representations of G existing at level k , one obtains, from the Kac-Peterson formula, the following relation between the matrix elements of S and the (classical) character polynomial:

$$S_{\lambda\mu}/S_{00} = \dim_q(\mu) \chi(\lambda)[\mu + \rho]. \quad (8.5)$$

The quantum dimension of μ is obtained as

$$\dim_q(\mu) = S_{\mu 0}/S_{00} = \chi(\mu)[\rho] = \chi(\mu; q^{2\rho^1}, q^{2\rho^2}, \dots, q^{2\rho^r})$$

where (ρ^j) are the components of the Weyl vector on the base of simple coroots (Kac labels), and $q = \exp(i\pi/(h^\vee + k))$. The previous relation for $S_{\lambda\mu}$ looks asymmetrical, but since S is symmetric, it implies

$$\dim_q(\mu) \chi(\lambda)[\mu + \rho] = \dim_q(\lambda) \chi(\mu)[\lambda + \rho]$$

Now, every sum rule for S (Theorem 3 or Theorem 4) leads immediately to a corresponding identity for the classical character polynomial. Using the symmetry property of S , the quantum dimension $\dim_q(\lambda)$ can be factored out, and we obtain the following property:

¹⁰We are very grateful to Patrick Dorey for refreshing our memory and for this nice observation.

Call X the Laurent polynomial $X(t_1, t_2, \dots, t_n) = \sum_{\mu \text{ with } \langle \theta, \mu \rangle \leq k} \chi(\mu)$,
then $X[\lambda + \rho] = 0$ if λ is of complex or quaternionic type.

Example: The character polynomials for the 6 irreps of $\mathfrak{su}(3)$ at level 2, of highest weights $\{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\}$ and of classical dimensions $\{1, 3, 3, 6, 8, 6\}$, are:

$$1, \frac{t_2}{t_1} + t_1 + \frac{1}{t_2}, \frac{t_1}{t_2} + \frac{1}{t_1} + t_2, \frac{t_2^2}{t_1^2} + t_1^2 + \frac{t_1}{t_2} + \frac{1}{t_1} + \frac{1}{t_2^2} + t_2, \\ \frac{t_1^2}{t_2} + \frac{t_2}{t_1^2} + \frac{t_1}{t_2^2} + \frac{t_2^2}{t_1} + t_1 t_2 + \frac{1}{t_1 t_2} + 2, \frac{t_1^2}{t_2^2} + \frac{1}{t_1^2} + \frac{t_2}{t_1} + t_1 + t_2^2 + \frac{1}{t_2}$$

The polynomial $X(t_1, t_2)$ is

$$3 + \frac{1}{t_1^2} + \frac{2}{t_1} + 2t_1 + t_1^2 + \frac{1}{t_2^2} + \frac{t_1}{t_2^2} + \frac{t_1^2}{t_2^2} + \frac{2}{t_2} + \frac{1}{t_1 t_2} + \frac{2t_1}{t_2} + \frac{t_1^2}{t_2} + 2t_2 + \frac{t_2}{t_1^2} + \frac{2t_2}{t_1} + t_1 t_2 + t_2^2 + \frac{t_2^2}{t_1^2} + \frac{t_2^2}{t_1}. \quad (8.6)$$

Its evaluation on the six h.w., using $q = \exp(i\pi/5)$, gives $\{\frac{3}{2}(3 + \sqrt{5}), 0, 0, 0, \frac{3}{2}(3 - \sqrt{5}), 0\}$.

8.4 On the path matrix X and its spectral properties

We use fusion matrices defined as $\hat{N}_\lambda = (\hat{N}_{\lambda\mu}^\nu)$. Using standard equalities $\hat{N}_{\lambda\mu}^\nu = \hat{N}_{\mu\lambda}^\nu$, $\hat{N}_{\lambda\mu}^\nu = \hat{N}_{\lambda\bar{\mu}}^{\bar{\nu}}$, and the conjugation matrix C introduced in sect. 1, with components $C_{\mu\nu} = \delta_{\mu\bar{\nu}}$, we have $\hat{N}_{\bar{\lambda}} = C\hat{N}_\lambda C$. Define the matrix $X = \sum_\lambda \hat{N}_\lambda$, dubbed “path matrix” for reasons explained below. From the corresponding property for \hat{N}_λ one obtains immediately $X = CXC$. The sum rule described by theorem 1 tells us that we can actually drop one of the two conjugation matrices in this equation. In other words, the equation $X = CX = CX$ holds. More generally, for any chosen module (nimrep) over the fusion algebra, one can define a path matrix $X = \sum_\lambda n_\lambda$ that enjoys similar properties.

There exist several interpretations of fusion coefficients (more generally of coefficients of nimreps) in terms of combinatorial constructions associated with fusion graphs: essential paths [24] (or generalizations of the latter), admissible triangles [25], (generalized) preprojective algebras or quivers [26], and they can also be used to define interesting weak Hopf algebras [27–29]. The translation of the sum rules involving the fusion coefficients (or those of the nimreps) into these different languages and points of view is left as an exercise to the reader. In the first combinatorial interpretation, the sum $\sum_{\nu\rho} \hat{N}_{\mu\nu}^\rho$ (or $\sum_{ab} n_{\mu a}^b$ for the nimreps) gives the dimension of the space of essential paths with fixed length μ , and matrix elements $X_{\nu\rho}$ (or X_{ab} for the nimreps) gives the dimension of the space of essential paths of arbitrary length, but with fixed origin and extremity. This explains the name “path matrix” given to X .

It is sometimes useful to consider, instead of S , the fusion character table $\chi = (\chi_{\mu\nu})$ with $\chi_{\mu\nu} = S_{\mu\nu}/S_{0\nu}$. The columns of that matrix are made of eigenvectors common to all fusion matrices (the first column giving the quantum dimensions of irreps), and the line labelled μ gives the corresponding eigenvalues for the fusion matrices N_μ (the first line being $1 \dots 1$). The matrix χ , in contradistinction to S , is not symmetric. Theorems 3 and 4 imply: $\sum_\mu \chi_{\mu\nu} = 0$ whenever ν is complex or quaternionic.

◆ 1 As all the \hat{N}_μ are diagonal in the same basis provided by the column-vectors $S[\nu]$ of S with eigenvalues $S_{\mu\nu}/S_{0\nu}$, see (1.5), their sum X has in the same basis the eigenvalues $\sum_\mu \chi_{\mu\nu}$. The

only possible non-zero eigenvalues of the path matrix X therefore correspond to irreps of real type. Example (continuation of (8.6)): The Lie algebra $\mathfrak{su}(3)$ at level 2 has 6 irreps, two of them being of real type (those of highest weight $(0,0)$ and $(1,1)$), the 6th degree characteristic polynomial of the corresponding path matrix X has therefore only two non vanishing roots.

◆ 2 From Verlinde formula it is easy to show that $\sum_{\mu'} \chi_{\mu\mu'} \chi_{\nu\mu'} = \text{Tr}(\hat{N}_\mu \hat{N}_\nu)$. In particular $\text{Tr}(\hat{N}_\mu) = \sum_\nu \chi_{\mu\nu}$. This sum over eigenvalues of a fusion matrix is automatically an integer. Warning: The numbers $\sum_\mu \chi_{\mu\nu}$ obtained previously as eigenvalues of X are usually not integers.

◆ 3 From the relation (8.5) between the S matrix and the classical character polynomials we obtain:

$$\chi_{\mu\nu} = \chi(\mu)[\nu + \rho].$$

Using the final result of section 8.3, the eigenvalues of the path matrix X , in particular its 0 eigenvalues, can be obtained from the evaluation of the Laurent polynomial (also called X there, on purpose), on the irreps that exist at the chosen level.

Example (continuation). The path matrix of $\mathfrak{su}(3)$ at level 2 is easily found to be $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

One can check that its non-zero eigenvalues are the two non-zero values obtained at the end of section 8.3.

◆ 4 Call $\mathfrak{s}_1 = \sum_\mu \dim_q(\mu)$ and $\mathfrak{s}_2 = \sum_\mu \dim_q(\mu)^2$. In section 1 we defined $\Sigma(\mu) = \sum_\lambda S_{\lambda\mu} = \dim_q(\mu) S_{00} \sum_\lambda \chi_{\lambda\mu}$. Using the standard result $\mathfrak{s}_2 = 1/S_{00}^2$, one finds $\Sigma(\mu) = \dim_q(\mu) (\sum_\lambda \chi_{\lambda\mu})/\sqrt{\mathfrak{s}_2}$. Since $\dim_q(\lambda) = \chi_{\lambda 0}$, we obtain in particular $\Sigma(0) = \mathfrak{s}_1/\sqrt{\mathfrak{s}_2}$.

Example (continuation). For an irrep $\lambda = (\lambda_1, \lambda_2)$ of $\mathfrak{su}(3)$ we can use the standard formula $\dim_q(\lambda) = (\lambda_1 + 1)_q (\lambda_2 + 1)_q (\lambda_1 + \lambda_2 + 2)_q / 1_q^2 2_q$, where $n_q = (q^n - q^{-n})/(q - q^{-1})$. By summing quantum dimensions (or their squares) over the Weyl alcove of level 2, we recover $\mathfrak{s}_1 = \frac{3}{2} (3 + \sqrt{5})$, that we have already obtained as the evaluation of the Laurent polynomial $X(t_1, t_2)$ on the 0 weight, or as one of the two non-zero eigenvalues of the path matrix X , and calculate $\mathfrak{s}_2 = \frac{3}{2} (5 + \sqrt{5})$. One finds $\Sigma(0) = \sqrt{3 + \frac{6}{\sqrt{5}}}$.

8.5 Final comments

Admittedly our proofs of Theorems 1-4 lack conciseness and more direct and conceptual proofs would be highly desirable. For example it is natural to wonder if there is a direct proof of Theorems 3 and 4, based on Galois arguments or some other hidden symmetry of the S matrix. If so, the proofs of Theorems 2 first (through Verlinde formula) and 1 then (through the large k limit) would follow.

Another tantalizing option would be to use Steinberg formula. Steinberg formula for tensor multiplicities reads $N_{\lambda\mu}^\nu = \sum_{v,w \in W} \text{sign}(vw) \mathcal{P}(v \cdot \lambda + w \cdot \mu - \nu)$ where $v \cdot \lambda = v[\lambda + \rho] - \rho$ is the Weyl shifted action, and \mathcal{P} is the Kostant partition function, which gives the number of ways one can represent a weight as an integral non-negative combination of positive roots. The highest weight of the conjugate of an irrep is the negative of the lowest weight of that irrep. The lowest weight is obtained from the highest weight by the action of the longest element¹¹ w_0 of the Weyl group. In

¹¹In all cases but A_{even} , $w_0 = \mathfrak{c}^{h/2}$ where \mathfrak{c} is a bipartite Coxeter element.

other words $\bar{\lambda} = -w_0[\lambda]$.

Our sum rule for tensor multiplicities therefore leads to various identities involving \mathcal{P} and w_0 . Conversely, a direct proof of such identities would provide a shorter derivation of theorem 1.

A The case of E_6

A.1 Sums of multiplicities for tensor products $\omega_{2,4} \otimes \mu$ of E_6

The detailed discussion of the tensor product of a representation of highest weight μ by one of the fundamental representations ω_2 or ω_4 of E_6 offers a good illustration of the three cases (i), (ii), (iii) presented in Section 2, and is anyway a mandatory step for the completion of our proof of Theorem 1. The aim of this appendix is to show how the cardinalities of the two classes (i) and (iii) and the total multiplicity may be proved to be the same for ω_2 and ω_4 .

For a given μ , and λ' one of the weights of the weight system $[\omega_2]$, we denote as before $\sigma = \lambda' + \mu + \rho$. Call $\phi^{\geq 0}$ the number of weights σ (counted with multiplicity) that have non-negative Dynkin labels. Those weights need not be Weyl reflected in the Racah–Speiser algorithm. Some of them, however, may lie on a wall of the fundamental Weyl chamber. Call ϕ_+^0 the number of the latter. The class (i) of weights with only positive Dynkin labels has thus cardinality $\phi = \phi^{\geq 0} - \phi_+^0$.

If one of the labels of σ is negative, we shall show below that a single Weyl reflection brings it back to the fundamental Weyl chamber, *including its walls*. Call $\psi^{\leq 0}$ the cardinality of that class, and ψ_-^0 the number of the reflected weights that lie on a wall of the fundamental Weyl chamber. The class (iii) of weights that contribute with a minus sign to the total multiplicity has cardinality $\psi = \psi^{\leq 0} - \psi_-^0$.

The total multiplicity is finally $\sum_{\nu} N_{\omega_2 \mu}^{\nu} = \phi - \psi = \phi^{\geq 0} - \psi^{\leq 0} - \phi_+^0 + \psi_-^0$. Note that $\phi^{\geq 0} + \psi^{\leq 0} = \phi + (\phi_+^0 + \psi_-^0) + \psi = 351$, the dimension of the ω_2 and ω_4 representations. All these numbers depend on the weight μ . What we want to prove is that for a given μ , $\sum_{\nu} N_{\omega_2 \mu}^{\nu} = \sum_{\nu} N_{\omega_4 \mu}^{\nu}$. In fact we shall establish that the numbers ϕ , ψ are the same for ω_2 and ω_4 .

- Let us first examine the $\psi^{\leq 0}$ weights σ that have a negative Dynkin label. As the weights λ' of the $[\omega_2]$ or $[\omega_4]$ systems have their labels equal to $0, \pm 1, \pm 2$, and at most one label equal to -2 , $\sigma_i = \lambda'_i + \mu_i + \rho_i = \lambda'_i + \mu_i + 1 \geq -1$; for a given σ , at most one Dynkin label σ_j equals -1 , and this requires $\lambda'_j = -2$ and $\mu_j = 0$. Conversely for each j such that $\mu_j = 0$, there are as many λ' fulfilling the above condition as there are weights λ' with $\lambda'_j = -2$. Both in the $[\omega_2]$ and the $[\omega_4]$ systems, this number is 15. Thus $\psi^{\leq 0} = 15 \times$ the number of vanishing labels $\mu_j = 0$ of μ .

We claim that any such σ with $\sigma_j = -1$ may be brought back to the fundamental Weyl chamber by a single Weyl reflection. To prove this point, take $\sigma = \sum_i \sigma_i \omega_i$ with all $\sigma_i \geq 0$ for $i \neq j$ and $\sigma_j = -1$. Then take the reflection s_j in the plane orthogonal to α_j : $s_j[\omega_i] = \omega_i - \delta_{ij} \alpha_j = \omega_i - \delta_{ij} \sum_{j'} \mathcal{C}_{jj'} \omega_{j'}$, with \mathcal{C} the Cartan matrix, hence $s_j[\omega_j] = -\omega_j + \sum_{j' \approx j} \omega_{j'}$, with the last sum running over the neighbours j' of j on the E_6 Dynkin diagram. This gives (as $\sigma_j = -1$)

$$\begin{aligned} s_j[\sigma] &= \sum_{i \neq j} \sigma_i \omega_i + \omega_j - \sum_{j' \approx j} \omega_{j'} \\ &= \omega_j + \sum_{j' \approx j} (\sigma_{j'} - 1) \omega_{j'} + \sum_{i \neq j, i \not\approx j} \sigma_i \omega_i. \end{aligned} \tag{A.1}$$

By inspection, one checks that if some λ' of $[\omega_2]$ or $[\omega_4]$ has $\lambda'_j = -2$, all the $\lambda'_{j'}$ for $j' \approx j$ are

non-negative, thus $\sigma_{j'} - 1 = \lambda'_{j'} + \mu_{j'} + \rho_{j'} - 1 \geq 0$ for $j' \approx j$, and for the other $i \neq j$, $i \not\approx j$ (neither j nor one of its neighbours), $\sigma_i = \lambda'_i + \mu_i + \rho_i \geq -1 + 0 + 1 = 0$, so that all labels of $w_j[\sigma]$ in (A.1) are non-negative, qed.

- Among these $\psi^{\leq 0}$ weights $s_j[\sigma]$ that have been reflected, ψ_-^0 have a vanishing Dynkin label. According to (A.1), this may happen only (a) if $\lambda'_{j'} = 0$ (and $\mu_{j'} = 0$) for some $j' \approx j$, or (b) if $\lambda'_i = -1$, $i \neq j$, $i \not\approx j$ (and $\mu_i = 0$). By inspection, one checks that for any node $j = 1, \dots, 6$ there exist 3 weights λ' in $[\omega_2]$ or in $[\omega_4]$ such that $\lambda'_j = -2$ and $\lambda'_{j'} = -1$ for each j' “neighbour” of j , thus 3 cases of type (a) per neighbour; and likewise one checks that there are four $\lambda' \in [\omega_2]$ or $\lambda' \in [\omega_4]$ satisfying condition (b) for each pair of (j, i) such that $\mu_j = \mu_i = 0$. Note that the fulfilment of these conditions is independent of the value of the non-vanishing labels of μ . There are, however, configurations where conditions (a) and/or (b) are satisfied for two pairs (j, j') or (j, i) , see an example below, and this depends on the detailed location of the vanishing labels of μ . We thus found more expedient to write a MathematicaTM code to enumerate the $62 = 2^6 - 2$ configurations of vanishing labels of $\mu \neq 0$, and for each of them, to count the number of λ' in $[\omega_2]$ or in $[\omega_4]$ that contribute to ψ_-^0 . As expected we found the same numbers for $[\omega_2]$ and $[\omega_4]$. We conclude that for a given μ , the number ψ of weights contributing negatively to the total multiplicity is the same for $[\omega_2]$ and $[\omega_4]$.

- We finally turn our attention to those weights that need not be reflected. Their number $\phi^{\geq 0} = 351 - \psi^{\leq 0}$ is the same for $[\omega_2]$ and $[\omega_4]$. It remains to count the number ϕ_+^0 that lie on one of the walls of the fundamental Weyl chamber. There too, it is easy to see that there is only a finite number of cases to consider. Indeed if σ has all its labels non-negative, $\sigma_j = 0$ occurs if $\lambda'_j = -2$ and $\mu_j = 1$, or if $\sigma_j = -1$ and $\mu_j = 0$. There are $\sum_{\ell=1}^6 \binom{6}{\ell} 2^\ell - 1 = 727$ choices for the labels of μ equal to 0 or 1, and one may write a code to check that for each of them, the number of λ' leading to a σ on a wall is the same for $[\omega_2]$ and $[\omega_4]$.

We conclude that $\phi = \phi^{\geq 0} - \phi_+^0$ is the same for $[\omega_2]$ and $[\omega_4]$, and so is $\sum_\nu N_{\omega_2/4\mu} \nu = \phi - \psi$, thus completing the proof of our assertion and of Lemma 1. \square

An explicit example

Let us illustrate the previous considerations on an explicit example. Take the weight $\mu = (1, 0, 0, 0, 2, 0)$. In the tensor product with ω_2 , there are $\phi^{\geq 0} = 351 - 4 \times 15 = 291$ weights $\sigma = \lambda' + \mu + \rho$ that have non-negative labels and thus belong to the fundamental chamber, and among them, $\phi = 38$ weights that do not belong to its walls. The corresponding weights $\lambda' + \mu$ give the following contribution to the tensor product :

$$\begin{aligned} & 5(0, 0, 0, 0, 2, 0) + 5(0, 0, 0, 0, 2, 1) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 1) + (0, 0, 1, 0, 2, 0) \\ & + 5(0, 1, 0, 0, 1, 0) + (0, 1, 0, 0, 1, 1) + (1, 0, 0, 0, 0, 1) + 5(1, 0, 0, 0, 3, 0) + 5(1, 0, 0, 1, 1, 0) \\ & + (1, 0, 1, 0, 0, 0) + (1, 1, 0, 0, 2, 0) + 5(2, 0, 0, 0, 1, 0) + (2, 0, 0, 0, 1, 1) \end{aligned}$$

Among the $\psi^{\leq 0} = 15 \times 4 = 60$ weights σ that could lead to a situation of type (iii), 39 lie on a wall; this 39 comes about in the following way: there are $3 + 3 \times 3 + 3 + 3 = 18$ cases of type (a) in the discussion above, coming from a $\lambda'_j = -2$ on node $j = 2, 3, 4, 6$, respectively, and $\lambda'_{j'} = 0$ on a node $j' \approx j$ with $j' \in \{2, 3, 4, 6\}$; and there are $4 \times 6 - 3$ cases of type (b), with 4 cases for each pair (j, i) of non neighbours taken in $\{2, 3, 4, 6\}$, but there is a double counting of three weights that fulfil (b) for two such pairs, for example, $(0, -1, 2, -1, 1, -2)$, whence the -3 ; and $18 + 24 - 3 = 39$. and there are thus only $\psi = 21$ that have no vanishing Dynkin label after reflection. For all these weights σ , $w[\sigma] - \rho$ therefore gives a negative contribution to the tensor product. One finds :

$$\begin{aligned} & 4(0, 0, 0, 0, 2, 0) + 3(0, 0, 0, 0, 2, 1) + 3(0, 1, 0, 0, 1, 0) + \\ & 4(1, 0, 0, 0, 3, 0) + 3(1, 0, 0, 1, 1, 0) + 4(2, 0, 0, 0, 1, 0) \end{aligned}$$

Subtracting the second contribution from the first, one gets the final result

$$\begin{aligned}\omega_2 \otimes \mu = & (0, 0, 0, 0, 2, 0) + 2(0, 0, 0, 0, 2, 1) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 1) + \\ & (0, 0, 1, 0, 2, 0) + 2(0, 1, 0, 0, 1, 0) + (0, 1, 0, 0, 1, 1) + (1, 0, 0, 0, 0, 1) + \\ & (1, 0, 0, 0, 3, 0) + 2(1, 0, 0, 1, 1, 0) + (1, 0, 1, 0, 0, 0) + (1, 1, 0, 0, 2, 0) + \\ & (2, 0, 0, 0, 1, 0) + (2, 0, 0, 0, 1, 1) .\end{aligned}$$

The total multiplicity is therefore $\phi - \psi = 38 - 21 = 17$.

If we now perform the same analysis for the tensor product $\omega_4 \otimes \mu$ with the same $\mu = (1, 0, 0, 0, 2, 0)$, we again get a positive contribution of 38 terms from the weights belonging to the fundamental chamber, and a negative contribution of 21, from the reflected weights, so that the total multiplicity, 17, is the same. It may be noticed that the obtained weights for $\omega_2 \otimes \mu$ and $\omega_4 \otimes \mu$ are quite different, both for the two contributions and for their sum. The final decomposition of $\omega_4 \otimes \mu$ reads as follows:

$$\begin{aligned}\omega_4 \otimes \mu = & (0, 0, 0, 0, 3, 0) + (0, 0, 0, 0, 3, 1) + 2(0, 0, 0, 1, 1, 0) + (0, 0, 0, 1, 1, 1) + \\ & (0, 0, 1, 0, 0, 0) + 2(0, 1, 0, 0, 2, 0) + (0, 1, 0, 1, 0, 0) + (1, 0, 0, 0, 1, 0) + \\ & 2(1, 0, 0, 0, 1, 1) + (1, 0, 0, 1, 2, 0) + (1, 0, 1, 0, 1, 0) + (1, 1, 0, 0, 0, 0) + \\ & (2, 0, 0, 0, 2, 0) + (2, 0, 0, 1, 0, 0)\end{aligned}$$

A.2 Sums of fusion coefficients in \hat{E}_6

Let us now see how the presence of the back wall affects the previous counting. We have to examine what happens to weights $\sigma = \lambda' + \mu + \rho$ that are either on or “beyond” the shifted back wall, *i.e.* have $\sigma_0 \leq 0$, and we have to see under which condition some reflected weight may lie on a wall (and hence not contribute to the multiplicity, according to the affine Racah–Speiser algorithm).

a. First consider any weight σ that undergoes a reflection as in sect. A.1. We prove that $s_j[\sigma]$ lies within the shifted principal alcove $\mathcal{K}(s_j[\sigma]) \leq k + h^\vee$, *including its back wall*. Here and in the following, j takes values in $\{1, 2, 3, 4, 5, 6\}$.

As in the previous section, we take σ with some $\sigma_j = -1$. By inspection, the weights λ' that have $\lambda'_j = -2$ have level $\mathcal{K}(\lambda') \leq 1$, hence $\mathcal{K}(\sigma) = \mathcal{K}(\lambda') + \mathcal{K}(\mu) + \mathcal{K}(\rho) \leq 1 + k + (h^\vee - 1) = k + h^\vee$, and for $s_j[\sigma] = \sigma - \langle \alpha_j, \sigma \rangle \alpha_j = \sigma - \sigma_j \alpha_j = \sigma + \alpha_j$, $\mathcal{K}(s_j[\sigma]) = \mathcal{K}(\sigma) + \mathcal{K}(\alpha_j)$. The levels of the simple roots α_j of E_6 are $(0, 0, 0, 0, 0, 1)$ for $j = 1, \dots, 6$. The previous inequality gives us $\mathcal{K}(s_j[\sigma]) \leq k + h^\vee$ for $j = 1, \dots, 5$, while s_6 , with the root α_6 of level 1, looks more problematic. Fortunately, one checks by inspection that all λ' with their 6-th Dynkin label equal to -2 have a level less or equal to 0, (reducing the previous bound by one unit), and thus we also have $\mathcal{K}(s_6[\sigma]) \leq k + h^\vee$.

b. We then turn to cases where σ is within the fundamental chamber but “beyond” the shifted back wall, *i.e.* has $\sigma_0 < 0$. Since

$$\sigma_0 = k + h^\vee - \mathcal{K}(\sigma) = (k - \mathcal{K}(\mu)) + (1 - \mathcal{K}(\lambda')), \quad (\text{A.2})$$

where the first bracket is non-negative, $\sigma_0 < 0$ occurs only for $\mathcal{K}(\lambda') = 2$ and $\mathcal{K}(\mu) = k$. Such a σ is brought back into the first (shifted) alcove by a single (affine) Weyl reflection: $s_0(\sigma) = \sigma - \theta$ whose level is indeed $\mathcal{K}(s_0(\sigma)) = k + h^\vee + 1 - \mathcal{K}(\theta) = k + h^\vee - 1$: $s_0[\sigma] - \rho$ has level equal to k and lies on the back wall of P_+^k . Now it is clear that the number of λ' of level equal to 2 is the same in the two (conjugate) weight systems $[\omega_2]$ and $[\omega_4]$.

c. We have finally to study cases where the unreflected weight σ or the reflected $s[\sigma]$ lies on a wall of the alcove. We leave aside the cases where the weight σ or $s_j[\sigma]$ lies on one of the ordinary walls of the Weyl chamber, that have been examined in the previous subsection, and we focus on cases where $s_0[\sigma]$ is on one of the ordinary walls, or where σ or $s_j[\sigma]$ is on the back wall.

In fact it is not difficult to show

- that the number of unreflected σ or reflected $s_j[\sigma]$ lying on the back wall of the fundamental alcove is the same for $[\omega_2]$ and $[\omega_4]$;
- that the number of reflected $s_0[\sigma]$ lying on one or several ordinary walls of the fundamental chamber is the same for $[\omega_2]$ and $[\omega_4]$.

This is proved as follows:

- As shown by (A.2), $\sigma \in P_+$ is on the (shifted) back wall, *i.e.* $\sigma_0 = 0$, iff μ is itself on the back wall ($\mathcal{K}(\mu) = k$) and λ' is of level 1. It is clear that for such a μ , the number of such λ' is the same in $[\omega_2]$ and $[\omega_4]$.
- Consider cases where σ has been reflected into $s_j[\sigma]$ which lies on the back wall of $P_+^{k+h^\vee}$. From the computation above, $\mathcal{K}(s_j[\sigma]) = \mathcal{K}(\sigma) + \mathcal{K}(\alpha_j) = \mathcal{K}(\sigma) + \delta_{j6}$ which may be equal to $k + h^\vee$ only if $\mathcal{K}(\sigma) = k$ and $\mathcal{K}(\lambda') = 1 - \delta_{j6}$. Now it is an easy matter to check that the number of λ' such that $\lambda'_j = -2$ and $\mathcal{K}(\lambda') = 1 - \delta_{j6}$ is the same in $[\omega_2]$ and $[\omega_4]$.
- Last we return to cases examined in b. above, where σ has to be reflected across the back wall, but assume now that $s_0[\sigma]$ is on an ordinary wall of the fundamental chamber. As seen above, $s_0[\sigma] = \sigma - \theta$, and $\theta = (0, 0, 0, 0, 0, 1)$ in the basis of fundamental weights, thus $(s_0[\sigma])_j = \sigma_j - \delta_{j6}$ which may vanish only for $j = 6$ and $\sigma_6 = 1$ (we have assumed that σ was not on an ordinary wall, otherwise it would have dropped out in section A.1, hence $\sigma_j > 0$). It is clear that for any given μ , the number of λ' such that $\lambda'_6 = -\mu_6$ is the same in $[\omega_2]$ and $[\omega_4]$. *qed*

The vanishing or negative contribution of these reflected weights to the sum of fusion coefficients is thus the same for ω_2 and ω_4 and we may finally conclude that

$$\sum_{\nu} \hat{N}_{\omega_2 \mu}^{\nu} = \sum_{\nu} \hat{N}_{\omega_4 \mu}^{\nu},$$

thus completing the proof of Theorem 2 for the \hat{E}_6 algebra.

An explicit example (continuation)

The reader can illustrate the above discussion with the following example. As in Appendix A1, we choose the irrep with highest weight $\mu = (1, 0, 0, 0, 2, 0)$ that exists at levels $k \geq 3$.

At level $k = 3$, *i.e.* $q^{15} = -1$,

$\omega_2 \otimes \mu = (0, 0, 0, 0, 2, 0) + (0, 0, 0, 1, 0, 0) + (0, 1, 0, 0, 1, 0) + (1, 0, 0, 0, 0, 1)$;
quantum dimensions of the r.h.s. are $\frac{1}{2}(5 + \sqrt{5})$, $\frac{1}{2}(5 + 3\sqrt{5})$, $\frac{1}{2}(5 + 3\sqrt{5})$, $\frac{1}{2}(5 + 3\sqrt{5})$.

$\omega_4 \otimes \mu = (0, 0, 0, 1, 1, 0) + (0, 0, 1, 0, 0, 0) + (1, 0, 0, 0, 1, 0) + (1, 1, 0, 0, 0, 0)$;
quantum dimensions of the r.h.s. are $2 + \sqrt{5}$, $\frac{3}{2}(1 + \sqrt{5})$, $\frac{3}{2}(3 + \sqrt{5})$, $2 + \sqrt{5}$.

Total dimension is $(\frac{1}{2}(5 + 3\sqrt{5})) \times (\frac{1}{2}(5 + \sqrt{5})) = 5(2 + \sqrt{5})$ in both cases, as it should.

Total multiplicity is 4 in both cases.

At level $k = 4$, *i.e.* $q^{16} = -1$

$\omega_2 \otimes \mu = (0, 0, 0, 0, 2, 0) + (0, 0, 0, 0, 4, 2) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 1) + (0, 2, 0, 0, 2, 0) + (1, 0, 0, 0, 0, 1) + (1, 0, 0, 0, 3, 0) + (2, 0, 0, 2, 2, 0) + (1, 0, 1, 0, 0, 0) + (2, 0, 0, 0, 1, 0)$

$\omega_4 \otimes \mu = (0, 0, 0, 0, 3, 0) + (0, 0, 0, 2, 2, 0) + (0, 0, 1, 0, 0, 0) + (0, 2, 0, 0, 4, 0) + (0, 1, 0, 1, 0, 0) + (1, 0, 0, 0, 1, 0) + (2, 0, 0, 0, 2, 2) + (1, 1, 0, 0, 0, 0) + (2, 0, 0, 0, 2, 0) + (2, 0, 0, 1, 0, 0)$

The reader can check that both r.h.s. have total quantum dimension $(3 + 2\sqrt{2} + \sqrt{2 - \sqrt{2}} + 3\sqrt{2 + \sqrt{2}}) \times (4 + 3\sqrt{2} + 2\sqrt{2 - \sqrt{2}} + 4\sqrt{2 + \sqrt{2}}) = 52 + 37\sqrt{2} + 4\sqrt{338 + 239\sqrt{2}}$

Total multiplicity is 10 in both cases.

B Automorphisms of affine algebras

We first describe these automorphisms for the algebras \hat{A}_n , $\hat{D}_{n=2s+1}$ and \hat{E}_6 which are used in section 4 for the proof of the vanishing of $\sum_{\lambda} S_{\lambda\kappa}$ when κ is complex. We then describe them for algebras \hat{B}_n , \hat{C}_n , $\hat{D}_{n=2s}$ and \hat{E}_7 which are used in section 5 that deals with the case where κ is quaternionic. There are no non-trivial automorphisms for \hat{F}_4 , \hat{G}_2 and \hat{E}_8 . These automorphisms reflecting the geometrical symmetries of the corresponding extended Dynkin diagrams are parametrized by the center of the chosen Lie group, or equivalently by classes of P/Q where P is the weight lattice, and Q is the root lattice. If ζ is an automorphism of the Weyl alcove we have $S_{\zeta(\lambda)\kappa} = \exp(\frac{2\pi i\tau(\kappa)}{N})S_{\lambda,\kappa}$ where τ is the corresponding character of the center, and N (sometimes called the connection index), the order of the center, is given by the determinant of the Cartan matrix. Automorphisms of affine algebras are explicitly listed in [4] but the values of τ , the corresponding character of the center, are not given there. The value of τ was calculated from the equality

$$\exp[\frac{2\pi i\tau(\kappa)}{N}] = \exp[-2\pi i\langle\kappa, f\rangle] \quad (\text{B.1})$$

where \langle, \rangle is the canonical bilinear symmetric form of the root space, and where f is an appropriate fundamental weight given as follows. Use the basic representation $f = \omega_1$ for A_n , B_n , E_6 , E_7 and $f = \omega_n$ for C_n . Use $f = \omega_n$ (one of the two spinorial irreps), $f = \omega_{n-1}$ (the other spinorial) and $f = \omega_1$, respectively for the three generators ζ' , ζ'' and ζ''' of the center $\mathbb{Z}_2 \times \mathbb{Z}_2$ of $D_{n=2s}$, each generator being equal to the product of the other two; finally, $f = \omega_n$ (one of the two spinorial irreps) for the given generator of $D_{n=2s+1}$.

$\hat{\mathfrak{g}}_k$	center of \mathfrak{g}	$\mathcal{K}(\lambda)$, generator(s), grading and conjugate:
\hat{A}_n	\mathbb{Z}_{n+1}	$\begin{aligned} \mathcal{K}(\lambda) &= \sum_{i=1}^n \lambda_i & \lambda_0 &= k - \mathcal{K}(\lambda) \\ \zeta(\lambda) &= (\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{n-1}) \\ \tau(\lambda) &= \sum_{i=1}^n i\lambda_i \pmod{n+1} \\ \lambda &= (\lambda_1, \dots, \lambda_n) \leftrightarrow \bar{\lambda} = (\lambda_n, \dots, \lambda_1) \end{aligned}$
$\hat{D}_{n=2s+1}$	\mathbb{Z}_4	$\begin{aligned} \mathcal{K}(\lambda) &= \lambda_1 + 2 \sum_{j=2}^{n-2} \lambda_j + \lambda_{n-1} + \lambda_n & \lambda_0 &= k - \mathcal{K}(\lambda) \\ \zeta(\lambda) &= (\lambda_n, \lambda_{n-2}, \lambda_{n-3}, \dots, \lambda_1, \lambda_0) \\ \tau(\lambda) &= 2 \sum_{\substack{j=1, \\ j \text{ odd} \\ n-2}}^{n-2} \lambda_j + \lambda_{n-1} + 3\lambda_n \pmod{4} & \text{if } n &= 1 \pmod{4} \\ \tau(\lambda) &= 2 \sum_{\substack{j=1, \\ j \text{ odd} \\ n-2}}^{n-2} \lambda_j + 3\lambda_{n-1} + \lambda_n \pmod{4} & \text{if } n &= 3 \pmod{4} \\ \lambda &= (\lambda_1, \dots, \lambda_{2s}, \lambda_{2s+1}) \leftrightarrow \bar{\lambda} = (\lambda_1, \dots, \lambda_{2s+1}, \lambda_{2s}) \end{aligned}$
\hat{E}_6	\mathbb{Z}_3	$\begin{aligned} \mathcal{K}(\lambda) &= \lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6 & \lambda_0 &= k - \mathcal{K}(\lambda) \\ \zeta(\lambda) &= (\lambda_0, \lambda_6, \lambda_3, \lambda_2, \lambda_1, \lambda_4) \\ \tau(\lambda) &= 2\lambda_1 + \lambda_2 + 2\lambda_4 + \lambda_5 \pmod{3} \\ \lambda &= (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \leftrightarrow \bar{\lambda} = (\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_6) \end{aligned}$

$\hat{\mathfrak{g}}_k$	center of \mathfrak{g}	generator(s) and grading:
\hat{B}_n	\mathbb{Z}_2	$\zeta(\lambda) = (k - \lambda_1 - 2 \sum_{i=2}^{n-1} \lambda_i - \lambda_n, \lambda_2, \lambda_3, \dots, \lambda_n)$ $\tau(\lambda) = \lambda_n \pmod{2}$
\hat{C}_n	\mathbb{Z}_2	$\zeta(\lambda) = (\lambda_{n-1}, \lambda_{n-2}, \dots, \lambda_2, \lambda_1, k - \sum_{i=1}^n \lambda_i)$ $\tau(\lambda) = \sum_{j \text{ odd}} \lambda_j \pmod{2}$
$\hat{D}_{n=2s}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\zeta'(\lambda) = (\lambda_{n-1}, \lambda_{n-2}, \dots, \lambda_2, \lambda_1, k - \lambda_1 - 2 \sum_{i=2}^{n-2} \lambda_i - \lambda_{n-1} - \lambda_n)$ $\tau'(\lambda) = 2 \sum_{j=1, j \text{ odd}}^{n-3} \lambda_j + 2\lambda_n \pmod{4}$ $\zeta''(\lambda) = (\lambda_n, \lambda_{n-2}, \dots, \lambda_2, k - \lambda_1 - 2 \sum_{i=2}^{n-2} \lambda_i - \lambda_{n-1} - \lambda_n, \lambda_1)$ $\tau''(\lambda) = 2 \sum_{j=1, j \text{ odd}}^{n-3} \lambda_j + 2\lambda_{n-1} \pmod{4}$ $\zeta'''(\lambda) = (k - \lambda_1 - 2 \sum_{i=2}^{n-2} \lambda_i - \lambda_{n-1} - \lambda_n, \lambda_2, \lambda_3, \dots, \lambda_{n-2}, \lambda_n, \lambda_{n-1})$ $\tau'''(\lambda) = 2\lambda_{n-1} + 2\lambda_n \pmod{4}$
\hat{E}_7	\mathbb{Z}_2	$\zeta(\lambda) = (k - \lambda_1 - 2\lambda_2 - 3\lambda_3 - 4\lambda_4 - 3\lambda_5 - 2\lambda_6 - 2\lambda_7)$ $\tau(\lambda) = \lambda_1 + \lambda_3 + \lambda_7 \pmod{2}$

Conventions: B_n has $n - 1$ long simple roots, the last root α_n is short. C_n has $n - 1$ short simple roots, the last root α_n is long. For E_7 , the root α_7 , at the extremity of the short branch is above the fourth vertex, counted from the left (this is not the convention of [4]).

C Types of representations for complex Lie groups and Lie algebras

C.1 A collection of known results

The following results are well known, see for instance [9, 10, 30], and are gathered here for the convenience of the reader.

\mathfrak{g}		<i>complex</i>	<i>self – conjugate</i>	
			<i>real</i>	<i>quaternionic</i>
A_n	$n = 1 \pmod 4$	$\mu_j \neq \mu_{n+1-j}$	$\mu_j = \mu_{n+1-j}$ $\mu_{(n+1)/2}$ even	$\mu_{(n+1)/2}$ odd
	$n \neq 1 \pmod 4$	$\mu_j \neq \mu_{n+1-j}$	$\mu_j = \mu_{n+1-j}$ always	never
B_n	$n = 0, 3 \pmod 4$	never	always	never
	$n = 1, 2 \pmod 4$	never	μ_n even always	μ_n odd
C_n		never	$\mu_1 + \mu_3 + \dots + \mu_m$ even $\mu_1 + \mu_3 + \dots + \mu_m$ odd $m = n$ if n is odd and $m = n - 1$ if n is even	
D_n	$n = 0 \pmod 4$	never	always	never
	$n = 2 \pmod 4$	never	$\mu_{n-1} + \mu_n$ even always	$\mu_{n-1} + \mu_n$ odd
	$n = 1, 3 \pmod 4$	$\mu_{n-1} \neq \mu_n$	$\mu_{n-1} = \mu_n$ always	never
E_6		$\mu_1 \neq \mu_5$ or $\mu_2 \neq \mu_4$	$\mu_1 = \mu_5$ and $\mu_2 = \mu_4$ always	never
E_7		never	$\mu_1 + \mu_3 + \mu_7$ even always	$\mu_1 + \mu_3 + \mu_7$ odd
E_8		never	always	never
G_2		never	always	never
F_4		never	always	never

C.2 Fusion and the Frobenius-Schur indicator

According to [11], see also [12], the second indicator I_μ of Frobenius-Schur, whose value is 1, 0 or -1 , according to the type (real, complex or quaternionic) of the representation μ of \mathfrak{g} , can be obtained as

$$I_\mu = \sum_{\nu\sigma} S_{0\sigma} \hat{N}_{\mu\nu}^\sigma S_{0\nu} \frac{\iota(\sigma)^2}{\iota(\nu)^2}$$

where $\iota(\nu) = \exp(2i\pi\mathfrak{h}(\nu))$ and $\mathfrak{h}(\nu) = \langle \nu, \nu + 2\rho \rangle / (\langle \theta, \theta \rangle (k + h^\vee))$ is the conformal weight of ν . It is not too difficult to show that $\iota(\nu) = T_{\nu\nu} \psi$ where $\psi = \exp(2i\pi c/24)$, $c = \dim(\mathfrak{g}) k / (k + h^\vee)$ is the central charge, and T is the modular matrix that obeys, together with S , the usual relations

$(ST)^3 = S^4 = 1$. Using the fusion matrix \hat{N}_μ , the previous relation between ι and T , the fact that S is symmetric, T is diagonal, $S^{-1} = SC$ and that $C_{0p} = \delta_{0p}$, we recast the formula giving the indicator as follows:

$$I_\mu = (S^{-1}TT\hat{N}_\mu T^{-1}T^{-1}S)_{00}$$

This last expression can be used to check easily the type of representations discussed in the text.

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